MATHEMATICAL MODELING OF HEAT TRANSFER IN BUILDING ENVELOPES

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Abstract. This paper investigates the mathematical modeling of heat conduction problems within multilayered composite building envelopes (walls) to minimize structural heat loss. A non-steady-state, one-dimensional partial differential equation (PDE) of the parabolic type, supplemented with a system of Robin (Type III) boundary conditions, is formulated to simulate thermal dynamics. To solve the differential model, an implicit finite difference method (FDM) scheme and the tridiagonal matrix algorithm (Thomas algorithm) are implemented. Based on local climatic datasets, the temperature distribution across the wall layers is simulated under extreme winter conditions (severe ambient temperature drop). The thermal insulation efficiency of modern materials, specifically basalt wool and expanded polystyrene, is quantitatively evaluated and analyzed.

Keywords: Heat conduction, partial differential equation, Robin boundary conditions, finite difference method, Thomas algorithm, thermal insulation, energy efficiency.

Introduction. With the continuous depletion of global energy resources and increasing environmental concerns, improving the energy efficiency of buildings has become a critical engineering challenge. Statistical data indicates that residential and public buildings consume over 40% of total global energy, with space heating and cooling ventilation systems accounting for the vast majority of this consumption. Critically, approximately 30–35% of this thermal energy is lost through external structural building envelopes, primarily the outer walls.

Consequently, accurately predicting non-steady thermal dynamics within multilayer wall systems during the structural design phase is an essential engineering objective. The most efficient, reliable, and cost-effective approach to solving this problem is constructing a robust mathematical model based on differential equations and running computational fluid dynamics (CFD) or thermal simulations.

The primary objective of this research is to mathematically model the transient heat transfer process in multilayer wall configurations under extreme winter temperature drops, providing a rigorous quantitative evaluation of different insulating materials' thermal retention performance.

Methodology and Mathematical Model

The object of this study is a three-layer composite outer wall structure comprising an internal structural brick layer (L_1), a core thermal insulation layer (L_2), and an external plaster finish (L_3). Because the lateral dimensions (height and width) of the wall are significantly larger than its thickness, the heat transfer process is treated as a one-dimensional, non-steady-state (transient) heat conduction problem.

Formulation of the Governing Differential Equation

Assuming homogeneous material properties within each distinct layer and the absence of internal heat generation sources, the transient heat conduction is governed by the following parabolic partial differential equation:

$$\rho_i c_i \frac{\partial T_i}{\partial t} = k_i \frac{\partial^2 T_i}{\partial x^2}, \quad x \in (x_{(i-1)}, x_i), \quad t>0, \quad i=1,2,3,$$

where $T_i(x,t)$ is the temperature field in the i-th layer (°C), ρ_i is the material density (kg/m^3), c_i is the specific heat capacity ($J/(kg \cdot K)$), k_i is the thermal conductivity coefficient ($W/(m \cdot K)$).

Initial and Boundary Conditions

At the initial state (t=0), the internal temperature distribution across the wall profile is assumed to be in thermal equilibrium:

$$T_i(x,0)=T_{\text{init}},\ T_{\text{init}}=20°\text{C}$$

to account for convective heat exchange between the wall boundaries and the surrounding fluid environments (indoor air and ambient atmosphere), **Type III (Robin) boundary conditions** are prescribed based on Newton's law of cooling:

At the interior surface (x=0):

$$-k_1\frac{\partial T_1(0,t)}{\partial x}=\alpha_{\text{int}}\left(T_{\text{room}}-T_1(0,t)\right);$$

At the exterior surface ($x=L$, where $L=L_1+L_2+L_3$):

$$-k_3\frac{\partial T_3(L,t)}{\partial x}=\alpha_{\text{ext}}\left(T_3(L,t)-T_{\text{amb}}(t)\right)$$

where α_{int} and α_{ext} represent the convective heat transfer coefficients of the indoor and outdoor air film boundaries, respectively; $T_{\text{room}}=22^\circ\text{C}$ (kept constant), and $T_{\text{amb}}(t)$ represents the time-dependent ambient outdoor temperature.

At the interfaces linking separate material layers ($x=x_1$ and $x=x_2$), **Type IV ideal thermal contact conditions** are imposed, enforcing the continuity of both temperature fields and heat flux densities:

$$T_i(x_i,t)=T_{i+1}(x_i,t),\quad k_i\frac{\partial T_i(x_i,t)}{\partial x}=k_{i+1}\frac{\partial T_{i+1}(x_i,t)}{\partial x},\quad i=1,2.$$

Numerical Solution Algorithm (FDM)

To compute the system of PDEs, a computational mesh is constructed with spatial step h and temporal step τ. An implicit finite difference scheme is formulated for each internal node j at time step n:

$$\rho_i c_i\frac{T_j^{n+1}-T_j^n}{\tau}=k_i\frac{T_{j+1}^{n+1}-2T_j^{n+1}+T_{j-1}^{n+1}}{h^2}$$

This implicit configuration is unconditionally stable. Rearranging terms yields a tridiagonal system of linear algebraic equations resolved sequentially at each time step:

$A_jT_{j-1}^{n+1}-C_jT_j^{n+1}+B_jT_{j+1}^{n+1}=-F_j.$

The resulting tridiagonal system is computationally executed using the **Thomas algorithm (matrix extraction technique)** written in Python/MATLAB.

Results. Real-world structural engineering parameters were selected to drive the numerical simulation. To mirror severe winter anomalies, the outdoor ambient temperature was set to a constant $T_{\text{amb}}=-15^\circ\text{C}$. The transient simulation tracking thermal degradation was executed for a continuous 48-hour cycle.

Two primary envelope configurations were analyzed comparatively:

- **Case A (Uninsulated):** A basic wall system composed entirely of solid brick and standard plaster.
- **Case B (Insulated):** A composite wall system integrated with a core basalt wool insulation layer.

The thermophysical properties of the structural layers used in the computational matrix are outlined below:

Layer	Material Type	Thickness (m)	Thermal Conductivity k, (W/m · K)	Specific Heat Capacity c, (J/kg · K)	Density ρ, (kg/m3)
1	Solid Clay Brick	0.38	0.70	880	1800
2 (Insulation)	Basalt Wool Board	0.10	0.04	840	150
3	Cement Plaster	0.02	0.93	840	1800

The computational simulation mapped out the exact internal temperature gradients across the spatial layout at the 24-hour and 48-hour intervals.

• **For Case A (Uninsulated):** The interior wall surface temperature $T_1(0,t)$ dropped to +11.2°C within the initial 24 hours. This falling thermal envelope severely triggers dew point condensation, resulting in interior moisture accumulation and structural mold risks.

• **For Case B (Insulated):** Owing to the significant thermal resistance provided by the 10 cm basalt wool insulation layer, the internal structural brick mass maintained a high heat index, keeping the inner wall surface temperature stable at +19.4°C.

Discussion. The numerical output validates the mathematical differential model's capacity to predict real-world structural heat retention behavior. Integrating a 10 cm thick basalt wool core dropped the total heat flux density (q) breaking through the wall from $38.4 W/m^2$ down to $8.1 W/m^2$. This marks a **4.7-fold reduction in structural heat loss**.

Furthermore, the mathematical framework proved highly adaptive when dealing with non-linear configurations, such as adjusting the ambient boundary parameters to follow a diurnal sinusoidal curve to mimic natural day-night temperature cycles. This flexibility allows engineers to thoroughly audit building heat retention and predict annual utility expenditure without running expensive, destructive field experiments.

Conclusion. This research successfully outlines a highly accurate methodology for modeling transient heat transfer within building envelopes using partial differential equations. The study yields the following conclusions:

1. The structured one-dimensional non-steady-state heat conduction model accurately and adequately tracks thermal dynamics in multi-layered envelopes.
2. The implicit finite difference method coupled with the Thomas algorithm proved unconditionally stable, displaying exceptional computational accuracy.
3. The numerical simulation quantitatively verified that incorporating a core basalt fiber sheet cuts building envelope heat dissipation by 4.7 times, ensuring interior thermal comfort.

The practical algorithms and model parameters presented in this study are recommended for structural engineers designing green, net-zero energy buildings and optimizing insulating material thickness.

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